Aside

→ Reduced MHD → { Reduced Representation for strong @ straight B_0 ↳ eliminates fast mode

Note: ① full MHD : 3 $\underline{V}$ components
2 $\underline{B}$ " " ($\nabla \cdot \underline{B} = 0$)
P, ρ

⇒ 7 components

② if $\nabla \cdot \underline{V} = 0$ ⇒ 4 components
($\rho = \text{const}$, P from $\nabla \cdot \underline{V} = 0$)

③ strongly magnetized system ⇒ Reduced MHD
⇒ scalar equations for ϕ, ψ (2 scalar fields)

Now:
- assume strong B_z (strong magnetization → gyrokinetics) ↳ later

"strong" ⟺ $\rho v^2 \sim p \ll B_z^2/8\pi$

so motion strongly anisotropic, and small scales generated in ⊥ direction only, as strong B_z inhibits line bending. (energy to perturb strong, high energy density field).

⇒ order : $B_z \sim \nabla_\perp \sim 1$

$B_\perp \sim \partial_z \sim O(\epsilon)$

Take $\rho \sim 1$, as $\underline{\nabla} \cdot \underline{V} = 0$ enforced by strong B_z;

$V_\perp^2 \sim p \sim B_\perp^2$ (i.e. equipartition of energy)
(springiness)

$\Rightarrow$ $V_\perp \sim \epsilon$, $p \sim \epsilon^2$, $\partial_t \sim \underline{V}_\perp \cdot \underline{\nabla}_\perp \sim \epsilon$

and pressure balance ($\underline{\nabla} \cdot \underline{V} = 0$ and incompressibility)

$\delta(B_z^2) \sim 2B_z \delta(B_z) \sim p$ (eqbrm)

i.e. $\omega \ll k(c_s^2 + V_A^2)^{1/2}$

$\Rightarrow \delta B_z \sim \epsilon^2$.

[idea is to order out the fast mode]

$\therefore$ to lowest order $\Rightarrow B_z = \text{const}$,

Now then: $(\underline{\nabla} \cdot \underline{B} = 0)$

$$\boxed{\begin{aligned} \underline{B} &= \hat{z} \times \underline{\nabla} \psi + B_z \hat{z} \\ &= \nabla A_\parallel \times \hat{z} + B_z \hat{z} \qquad \psi = -A_\parallel \end{aligned}}$$

B rep. by single scalar potential

$\underline{\nabla} \cdot \underline{B} = \partial_z \tilde{B}_z = \epsilon^3 \Rightarrow 0$.

parallel comp. of vector pot.

Similarly,

$\partial_z p \sim O(\epsilon^3)$, $\Rightarrow$ $\cancel{V_z} \ll V_\perp$
$J_\perp B_\perp \sim \epsilon^3$ neglect V_z.

Now, $\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi = -\frac{\underline{v}\times\underline{B}}{c}$

$\Rightarrow \quad +\frac{1}{c}\frac{\partial \underline{A}}{\partial t} = \frac{\underline{v}\times\underline{B}}{c} - \underline{\nabla}\phi \qquad (*)$

$B_z = (\underline{\nabla}_\perp \times \underline{A}_\perp)\cdot\hat{z}$

so $\partial_t A_\perp \sim \epsilon^3$ (a la $\partial_z \rho_z$)

$\therefore \quad \nabla_\perp \phi \cong \left(\frac{\underline{v}\times\underline{B}}{c}\right)_\perp$, in $(*)$

$\Rightarrow \quad \boxed{\underline{V}_\perp = \frac{c\,\hat{z}\times\underline{\nabla}\phi}{B_z}}$ $\perp$ velocity

$\rightarrow$ $\left\{\begin{array}{l}\text{motion } \perp \text{ is} \\ \underline{E}\times\underline{B}\end{array}\right.$.

Now, taking parallel component of $(*)$.
(units!)

$\Rightarrow \quad \frac{\partial}{\partial t}\psi + \underline{v}\cdot\underline{\nabla}\psi = B_z \partial_z \phi$

so have $\left(\begin{array}{c}\text{vector potential} \\ \text{flux}\end{array}\right)$ equation:

$$\boxed{\frac{\partial \psi}{\partial t} + \underline{v}\cdot\underline{\nabla}\psi = B_z\,\partial_z\phi}$$

$$= B_z\hat{z} + \hat{z}\times\nabla\psi$$

or, alternatively, $$\boxed{\frac{\partial\psi}{\partial t} - \underline{B}\cdot\nabla\phi = 0}\,.$$

Finally, for ϕ, write:

$\perp$ motion

cells of $E\times B$ drift, ('spin up' rate?)

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v} = \frac{-\nabla p}{\rho_0} + \frac{\underline{J}\times\underline{B}}{c}$$

$(\nabla\times)\cdot\hat{z} \Rightarrow$ vorticity component $(\parallel \hat{z})$ evolution

$$\frac{\partial}{\partial t}\omega_z + \underline{v}_\perp\cdot\nabla\omega_z = \cancel{-\nabla\times\frac{\nabla p}{\rho_0}}^{\,0} + \hat{z}\cdot\nabla\times\left(\frac{\underline{J}\times\underline{B}}{c}\right)$$

$$= \underline{B}\cdot\nabla J_z - \cancel{\underline{J}\cdot\nabla B_z} \qquad \delta B_z \sim \epsilon^3$$

$$\cong \underline{B}\cdot\nabla J_z$$

$\therefore$ $$\boxed{\frac{\partial}{\partial t}\omega_z + \underline{v}\cdot\nabla\omega_z = \underline{B}\cdot\nabla J}$$

but:

$$\omega_z = \hat{z}\cdot\nabla\times\underline{v} = \nabla^2\phi$$

$$J_z = \hat{z}\cdot(\nabla\times\underline{B})\frac{c}{4\pi} = \nabla^2\psi$$

so finally have:

$$\boxed{\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi = B_z\frac{\partial}{\partial z}\nabla^2\psi + \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi}$$

Finally, have reduced MHD equation:

$$\frac{\partial\psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\psi = B_z\,\partial_z\phi + \eta\nabla^2\psi$$

$$\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi - \nu\nabla^2\nabla^2\phi = \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi + B_z\frac{\partial}{\partial z}\nabla^2\psi$$

- note have reduced MHD to $\underline{\underline{2}}$ <u>scalar evolution equations</u>

- does this look familiar?

even stronger!

- for 2D MHD:

$$\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi = \underline{B}\cdot\underline{\nabla}\,\nabla^2\psi + \nu\nabla^2\nabla^2\phi$$

$$\frac{\partial\psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\psi = \eta\nabla^2\psi$$

- ① Conservation Laws, etc. (HW)

$$\frac{d}{dt}E = \underset{(\text{to } \eta, \nu)}{0}, \quad E = \int d^3x \left[\frac{(\nabla\phi)^2}{2} + \frac{(\nabla\psi)^2}{2}\right]$$

② $\mathcal{H} = A\cdot B \cong \underset{\substack{\uparrow \\ \text{const.}}}{B_z}\psi$

$\Rightarrow \quad H = \int d^3x\, B_z\psi \quad , \quad \frac{dH}{dt} = 0 \; , \text{ to } O(\eta)$

Ohm's Law (flux advection) is simple statement of helicity conservation.

? $\frac{\partial}{\partial t}\psi + \nabla\cdot\Gamma_\psi = \eta\nabla^2\psi$ form Γ_ψ s/t $\begin{cases} H \text{ conserved} \\ E_M \text{ dissipated} \end{cases}$

③ $K = \int d^3x\, \underline{V}\cdot\underline{B} = \int d^3x\, (\nabla\phi\cdot\underline{\nabla}\psi)$

also conserved, to dissipation.

→ General Intro
→ Suydam Limit
→ Resistive Interchange

.) Interchange Dynamics in Sheared Magnetic Field

Now in "cylindrical" tokamak : $(\epsilon = a/R \ll 1)$

- $\underline{B}_0 = B_\theta(r)\hat{\theta} + B_z\hat{z}$ $\quad |B_\theta| < B_z$

- periodic perturbations ⇒

$$\hat{\phi} = \sum_{m,n} \hat{\phi}_{m,n}\, e^{i(m\theta - n\phi)}$$

$\theta \equiv$ poloidal ∡
$\phi \equiv$ toroidal ∡

Then, note:

$$\underline{B}\cdot\underline{\nabla} = \frac{B_\theta(r)}{r}\frac{\partial}{\partial\theta} + \frac{B_z}{R}\frac{\partial}{\partial\phi}$$

$$\to i\left(m\frac{B_\theta(r)}{r} - \frac{n}{R}B_z\right)$$

$$= i\frac{B_z}{R}\left(\frac{m}{q(r)} - n\right)$$

$q(r) = rB_z / RB_\theta \equiv$ local pitch of magnetic field ("safety factor")

Thus, $k_\parallel^{m,n} = \frac{1}{R}\left(\frac{m}{q(r)} - n\right)$

– tends to be small (i.e. line bending, Landau damping, etc. weak) when

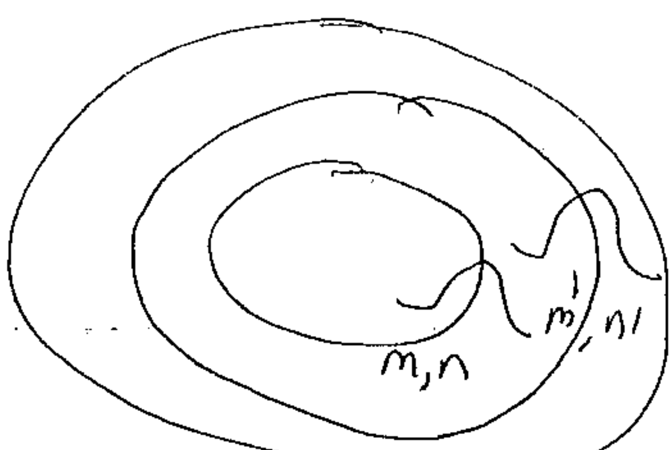
$q(r) = m/n$ i.e. $\left\{\begin{array}{l}\text{local pitch of field line} \\ = \text{pitch of perturbation}\end{array}\right.$

– defines $r_{m,n}$ s/t $q(r_{m,n}) = m/n$

i.e. $r_{m,n}$ is radius of $\left\{\begin{array}{l}\text{mode rational surface} \\ \text{resonant surface}\end{array}\right.$

where mode naturally wants to sit, to minimize bending, dissipation etc.

i.e.



Fluctuations in tokamaks tie to resonant surf.

natural to write $\vec{\phi}_{m,n} = \vec{\phi}_{m,n}(x)$

where $x = r - r_{m,n}$

Note: $k_{\parallel} = \frac{1}{R}\left(\frac{m}{q(r_{m,n}+x)} - n\right)$

$$= \frac{1}{R}\left(-\frac{m q'_{m,n} x}{q^2_{m,n}}\right) + \text{h.o.t}$$

$$\equiv \frac{k_\theta x}{L_s} \qquad k_\theta = m/r$$

$$\frac{1}{L_s} = -\frac{r q'}{R q^2} \equiv \text{magnetic shear length}$$

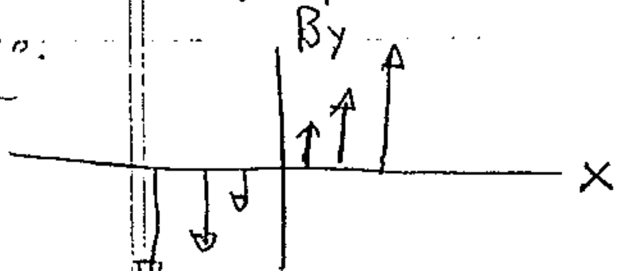
- equivalent to placing a resonant mode in local field

$$\underline{B} = B_0\left(\hat{z} + \frac{x}{L_s}\hat{y}\right) \equiv \text{sheared slab model.}$$

Now, can furthur observe:

- in sheared system, field lines have radially varying orientation

i.e.:



$B_y = B_0 x / L_s$

to interchange two flux tubes, need rotate (i.e. frozen in) flux tubes to align (locally) with sheared field

$\Rightarrow$ expect sheared field will exert significant stabilizing effect in ideal interchange.

i.e. $\frac{1}{c}\frac{\partial}{\partial t}\hat{A} = \frac{B_z}{4\pi}\nabla_{\parallel}\vec{\phi}$, $J_z = -\nabla_\perp^2 A$

$$-\frac{\partial^2}{\partial t^2}\nabla_\perp^2\vec{\phi} = \frac{\underline{B_0}\cdot\nabla}{\rho_0 c}\frac{\partial \vec{J}_z}{\partial t} + \frac{\partial^2\vec{\phi}}{\partial y^2}\frac{g_{eff}}{L_\rho}$$

$$\hat{\phi} = \sum_{m,n} e^{\gamma_{m,n}t}\,\vec{\phi}_{m,n}(x)\,e^{i(m\theta - n\phi)}$$

$$+\gamma^2\left(\frac{\partial^2}{\partial x^2} - k_\theta^2\right)\vec{\phi}_{m,n} = +\frac{\cancel{\gamma} B_z i k_\parallel}{\rho_0 \cancel{c}}\frac{\nabla_\perp^2}{\cancel{\gamma}}\left(\frac{\cancel{c} B_z i k_\parallel \vec{\phi}_{m,n}}{4\pi}\right) + k_\theta^2\frac{g_{eff}}{L_\rho}\vec{\phi}_{m,n}$$

$$\gamma^2\left(\frac{\partial^2}{\partial x^2} - k_\theta^2\right)\vec{\phi}_{m,n} = -V_A^2 k_\parallel \nabla_\perp^2\left(k_\parallel\vec{\phi}_{m,n}\right) + k_\theta^2\frac{g_{eff}}{L_\rho}\vec{\phi}_{m,n}$$

$$\gamma_{m,n}^2 = \frac{\left[-k_\theta^2\frac{g_{eff}}{L_\rho}\int|\vec{\phi}_{m,n}|^2 dx - V_A^2\int dx\,|\nabla_\perp k_\parallel\vec{\phi}_{m,n}|^2\right]}{\int|\nabla_\perp\hat{\phi}|^2 dx}$$

..For scaling:

$\nabla_\perp \sim 1/a$ (key: No scale for $\vec{\phi}$, other than a!)

$k_\parallel \sim \frac{k_\theta x}{L_s} \sim \frac{k_\theta a}{L_s}$

So, for β_{crit} (transition to instability):

$$\frac{g_{eff}}{|L_p|} \geq \frac{V_A^2}{L_s^2} \Rightarrow \frac{L_s^2}{|L_p| R_c} \beta_{crit} > 1$$

stability if $\beta \leq \frac{|L_p| R_c}{L_s^2} \sim O(\epsilon^2)$ as $L_s \sim R$.

i.e. shear forces line-tying effect via $\nabla_\parallel \Rightarrow \sim 1/L_s$.

More detailed analysis confirms basic scaling
$\beta \leq \frac{L_p R_c}{L_s^2}$ (Suydam limit).

Now, useful to consider resistive interchange in sheared field

- allows field, fluid to slip (not frozen in!)

- introduces small scale $\Delta x \sim (\eta/\omega)^{1/2} \ll a$

→ Suydam Done Simply ...

Can write reduced MHD equations:

① $$m_i n_0 \frac{d}{dt} \nabla_\perp^2 \phi = \frac{1}{c}(\underline{B}\cdot\underline{\nabla}) J_\parallel - \frac{K}{r}\frac{\partial p}{\partial \theta}$$

$K \equiv 2B_\theta^3 / r B^2 \sim 1/R_c \equiv$ curvature of field lines

crucial - Suydam is stability limit for ideal interchange.

and ② $\hat{E}_\parallel = 0$

③ $$\frac{dp}{dt} = 0, \qquad \text{as } \underline{\nabla}\cdot\underline{V} = 0$$

So, can immediately write:

$$\omega m_i n_0 \nabla_\perp^2 \hat{\phi} = -\frac{B_0}{4\pi r}(m - nq)\nabla_\perp^2 \tilde{\psi} + \frac{m}{rc}\langle J_\parallel \rangle' \tilde{\psi}$$

$$+ \frac{2m}{q^2 R^2}\tilde{p}$$

$$\omega \tilde{\psi} = -\frac{B_0}{r}(m - nq)\tilde{\phi}$$

$$\omega \tilde{p} = -\frac{m}{r}\tilde{\phi}\langle p \rangle'$$

so, can assemble as:

$$\omega m_i n_0 \nabla_\perp^2\left(\frac{\omega\tilde{\psi}}{-\frac{B_\theta}{r}(m-n\epsilon)}\right) = -\frac{B_\theta}{4\pi r}(m-n\epsilon)\nabla_\perp^2\tilde{\psi}$$

$$+\frac{m}{rc}\langle J_\parallel\rangle'\,\tilde{\psi} + \frac{2m}{\epsilon^2R^2}\left(\frac{-m}{\omega r}\hat{\phi}\frac{dp_0}{dr}\right)$$

but $\hat{\phi} = \omega\tilde{\psi}\Big/-\frac{B_\theta}{r}(m-n\epsilon)$

$$\Rightarrow \omega^2\left\{m_i n_0\nabla_\perp^2\left(\tilde{\psi}\Big/\frac{B_\theta}{r}(m-n\epsilon)\right)\right\} = -\frac{B_\theta}{4\pi r}(m-n\epsilon)\nabla_\perp^2\tilde{\psi}$$

$$+\frac{m}{cr}\langle J_\parallel\rangle'\,\tilde{\psi} + \frac{2m}{\epsilon^2R^2}\left(\frac{m}{\cancel{\omega r}}\frac{dp_0}{dr}\,\frac{\cancel{\omega}\tilde{\psi}}{-\frac{B_\theta}{\cancel{r}}(m-n\epsilon)}\right)$$

Now, if seek determine marginality criterion, take $\omega^2 \to 0$, so:
(exchange of stabilities!)

$$\boxed{\underset{①}{\nabla_\perp^2\tilde{\psi}} = \underset{②}{\frac{4\pi m}{(m-n\epsilon)cB_\theta}\langle J_\parallel\rangle'\,\tilde{\psi}} + \underset{③}{\frac{8\pi m^2}{B_T^2(m-n\epsilon)^2 r}\frac{dp_0}{dr}\,\tilde{\psi}}}$$

→ Above is Newcomb Equation → equation for marginal displacement (i.e. equiv. to Euler eqn) in ideal MHD i.e.
↳ $\frac{\partial \xi_r}{\partial t} = 0$ ⇒ "perturbed eqbrm"

$$\begin{cases} \frac{1}{c}(\underline{B}\cdot\underline{\nabla})J_\parallel - \frac{2}{\varepsilon^2 R^2}\frac{\partial p}{\partial \theta} = 0 \\ \text{with } \underline{B}\cdot\underline{\nabla}p = 0 \end{cases} \Rightarrow i k_\parallel \tilde{p} = -\frac{\tilde{B}_r}{B_0}\frac{\partial}{\partial r}p_0$$

$$\Rightarrow \underset{①}{\frac{1}{c} i k_\parallel \nabla^2\tilde{\psi}\frac{c}{4\pi}} + \underset{②}{\frac{1}{c}\tilde{B}_r\frac{\partial\langle J_\parallel\rangle}{\partial r}} - \underset{③}{\frac{2}{\varepsilon^2 r^2} i m\left(-\frac{\tilde{B}_r}{B_0 i k_\parallel}\frac{dp_0}{dr}\right)} = 0$$

① current perturbation $\tilde{J}_\parallel$

② displacement of eqbrm. current → drives kinks

③ curvature driven current ("Pfirsch-Schluter") → drives interchanges

→ Obviously, Newcomb equation fails at $x \to 0$, unless $\hat{\psi} \to 0$, on rational surfaces. Need dynamics, inertia, etc. or $\left\{ \begin{array}{l} \text{resistivity} \\ \text{nonlinearity} \ldots \end{array} \right.$

Now:

$$\frac{4\pi m \langle J_{\parallel} \rangle'}{(m-nq)cB_0} = \frac{4\pi\, m/n \langle J_{\parallel} \rangle'}{\left(\frac{m}{n}-q\right)cB_0} = \frac{4\pi q \langle J_{\parallel} \rangle'}{-q' x c B_0} \equiv \frac{\delta}{x}$$

$$\frac{8\pi m^2}{B^2(m-nq)^2 r}\frac{dp_0}{dr} \equiv \frac{-\gamma}{x^2}$$

$\boxed{\gamma > 0}$

$\gamma = \dfrac{-8\pi r\, dp_0/dr}{B^2 \hat{s}^2}$

$\hat{s} = rq'/q$ shear parameter $\Rightarrow$ rate of pitch rotation.

$\therefore$ have:

$$-\nabla_{\perp}^2 \psi + \frac{\delta}{x}\psi - \frac{\gamma}{x^2}\psi = 0$$

as interested in pressure driven modes (i.e. interchanges), take $\delta = 0$.

$$\therefore \quad \left(\nabla_r^2 - \frac{m^2}{r^2} + \frac{\gamma}{x^2}\right)\psi = 0$$

$$k_r^2 \gg k_\theta^2 \Rightarrow \psi \sim x^{\nu}$$

$$\nu(\nu-1) + \gamma = 0$$

$r^2 - r + \gamma = 0$

$r = \frac{1}{2} \pm \frac{1}{2}(1-4\gamma)^{1/2}$

$\therefore$ to avoid instability, need avoid nodes, so

$\boxed{\gamma < 1/4}$ → recovers Suydam criterion. (see 10a)

i.e. $\boxed{-\frac{8\pi r}{B^2 \hat{S}^2}\frac{dp_0}{dr} < 1/4}$ → limit on pressure gradient due shear

→ Physics of Suydam Criterion

Note can write:

$r \, -\frac{dp_0}{dr}\frac{4\pi}{B^2} < \frac{\hat{S}^2}{8}$

$$\Rightarrow \boxed{\frac{r}{L_p}\beta < \frac{\hat{S}^2}{8} = \left(\frac{r q'}{q}\right)^2 / 8}$$

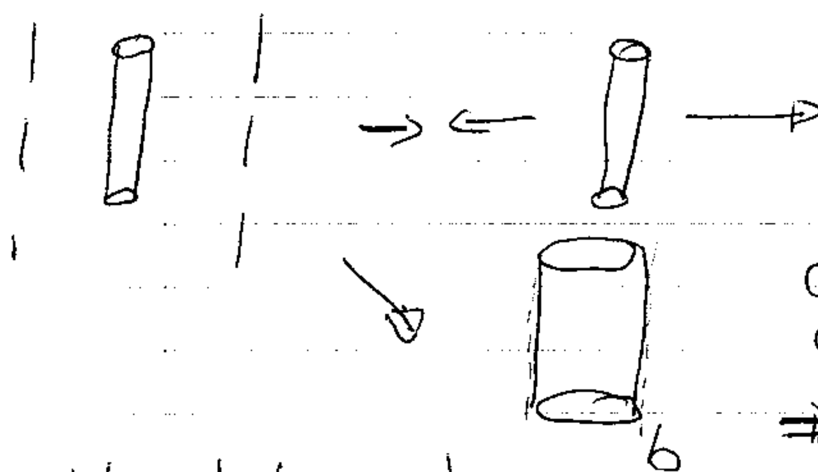
$r q'/q = (1/L_s^2)(qR)^2$

⇒ β-limit, ∇p-limit criterion (in terms stability)

stability $\Leftrightarrow$ eigenstructure

b.) Pinch – $\left\{ \begin{array}{l} \text{Distributed Current} \\ \text{Localized Modes / Resonant} \end{array} \right.$

Now $\rightarrow$ opposite extreme simplifacation from surface current pinch



conducting wall up/against plasma $\Rightarrow$ no gap, etc.

$$\Rightarrow \delta W = \underbrace{\frac{1}{2}\int_{V_i} d^3x}_{\text{internal term only}} \Big[\gamma p_0 (\underline{\nabla}\cdot\underline{\xi})^2 + \frac{1}{4\pi}(\nabla\times\underline{\xi}\times\underline{B}_0)^2$$

$$+ \underline{\xi}\cdot\underline{\nabla} p_0\, \underline{\nabla}\cdot\underline{\xi} - \frac{1}{4\pi}(\xi\times\underline{\nabla}\times\underline{B}_0)\cdot(\underline{\nabla}\times\underline{\xi}\times\underline{B}_0) \Big]$$

furthur, assume cylindrical/helical symmetry $\Rightarrow$

$$\underline{\xi} = \underline{\xi}(r)\, e^{i(m\theta + kz)}$$

$$\frac{\delta W}{\delta \mathcal{E}_\theta} = 0 \Rightarrow \frac{m}{r}\tilde{\mathcal{E}}_\theta + k\tilde{\mathcal{E}}_z = \frac{i}{r}\frac{d}{dr}(r\tilde{\mathcal{E}}_r)$$

$$\frac{\delta W}{\delta \mathcal{E}_z} = 0 \Rightarrow B_z\tilde{\mathcal{E}}_\theta - \tilde{\mathcal{E}}_z B_\theta =$$

$$= \frac{-i}{k^2r^2+m^2}\left[(krB_\theta - mB_z)\frac{d\mathcal{E}_r}{dr} - (krB_\theta + mB_z)\frac{\mathcal{E}_r}{r}\right]$$

so, can eliminate $\mathcal{E}_\theta$, $\mathcal{E}_z$ and write (after I. B. A.):

$$\boxed{\delta W = \frac{\pi}{2}\int_0^b dr\left\{f\left(\frac{d\mathcal{E}_r}{dr}\right)^2 + g\,\mathcal{E}_r^2\right\}}$$

1D system!

$$f = \frac{r}{4\pi}\frac{(krB_z + mB_\theta)^2}{k^2r^2+m^2} = \frac{r}{4\pi}\frac{(\underline{k}\cdot\underline{B})^2}{(k^2+m^2/r^2)}$$

$$g = \frac{2k^2r^2}{k^2r^2+m^2}(dP/dr) + \frac{r}{4\pi}(\underline{k}\cdot\underline{B})^2\frac{\left(k^2+\frac{m^2}{r^2}-1/r^2\right)}{(k^2+m^2/r^2)}$$
$$+\left(2k^2r^3/4\pi(k^2r^2+m^2)^2\right)\left(k^2B_z^2 - m^2B_\theta^2/r^2\right)$$

Now, $\frac{\delta W}{\delta \xi_r} = 0 \Rightarrow$

$$\boxed{\begin{aligned} &\frac{d}{dr} f \frac{d\xi_r}{dr} - g\,\xi_r = 0 \\ &\xi_r \big|_b = 0 \\ &\xi_r \big|_0 \text{ finite} \end{aligned}}$$

(E.O.M.) "equation of motion" for displacement → marginal

Now, can furthur comment:

→ full solution is extremum of
$L = T - W$

$\therefore\ \omega^2 \neq 0 \Rightarrow g \to g + \underbrace{g_1}_{\text{extra term}} \begin{cases} > 0 \text{ for } \omega^2 < 0 \\ < 0 \text{ for } \omega^2 > 0 \end{cases}$

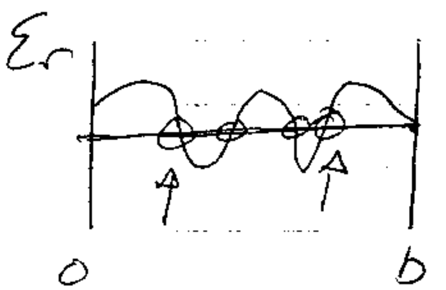
$$\left(\begin{aligned} L &= +\omega^2 |\tilde{\xi}|^2 - W \\ -L &= W - \omega^2 |\tilde{\xi}|^2 \end{aligned} \right)$$

(i.e. $g < 0$)

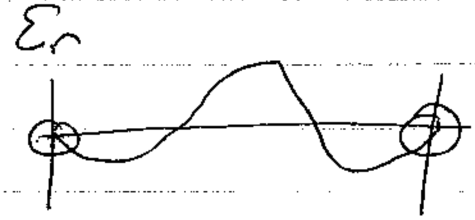
→ assume solution of E.O.M.
has more than two zeroes in $(0, b)$.

$\therefore$ by adding $g_1 > 0$ ($\Rightarrow \omega^2 < 0$)

can shift zeroes:



adding $\oplus$ to g $\Rightarrow$



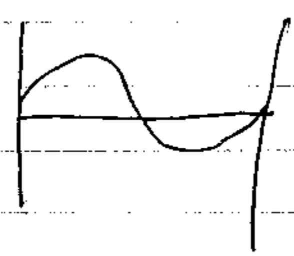
$$\left\{\begin{array}{l} \dfrac{d}{dr} f \dfrac{d\mathcal{E}_r}{dr} - g\,\mathcal{E}_r = 0 \\ \Rightarrow \mathcal{E}_r \text{ wiggles } \underline{\text{less}}! \end{array}\right.$$

(wiggles $\Rightarrow$ $g < 0$)

$\Rightarrow$ modified solution satisfies boundary conditions!

$\therefore$ corresponds to <u>unstable</u> solution. $\omega^2 < 0$.

<u>but</u> if solution E.O.M. has fewer than two zeroes (i.e. 1 zero):



, can only satisfy b.c.'s by $\mathcal{E}_r$ wiggling <u>more</u>.

$\Rightarrow$ must add <u>negative</u> g, to g $\Rightarrow$ $\omega^2 > 0$.

⇒ Why care about this?

⇒ if more than two zeroes in ξ_r solving E.O.M. ⇒

instability!

⇒ if fewer, stability

∴ establishes connection between oscillations/structure of E.O.M. solution and stability.
⇒ stability ⇔ avoid oscillations.

Implications ⇒

- consider resonant, large m mode

- now, want $\delta W < 0$ for instability

but: $g = \dfrac{r(k\cdot B)^2}{4\pi} \dfrac{\left(k^2 + \frac{m^2}{r^2} - \frac{1}{r^2}\right)}{\left(k^2 + m^2/r^2\right)} + \cdots$

and: $m \to \infty$

⇒ $\underline{k} \cdot \underline{B}$ must $\to 0$

– structurally similar to line-tied inter-change criterion,

i.e. schematically

$$\gamma^2 = \gamma_I^2 - k_\parallel^2 V_A^2 \quad (n \to \infty)$$

$$= \cancel{\frac{k_\theta^2}{k_\perp^2}}\, \frac{\kappa}{p} \frac{dp}{dr} c_s^2 - k_\parallel^2 V_A^2$$

$$= \frac{c_s^2}{R_c L_p} - k_\parallel^2 V_A^2$$

Now, in sheared system, with resonances

$$k_\parallel = \frac{k_\theta x}{L_s} \sim \frac{k_\theta \Delta x}{L_s},$$ If take $(\Delta x) k_\theta \sim 1$
i.e. no other scale ... in ideal MHD

$$\gamma^2 = \frac{c_s^2}{R_c L_p} - \frac{V_A^2}{L_s^2}$$

$\Rightarrow$ stability for $\dfrac{(c_s/V_A)^2}{R_c L_p} < \dfrac{1}{L_s^2}$

if take $1/L_s = \hat{s}/R_L$

$$\Rightarrow \frac{R^2 q^2}{R_c L_p} \beta < \hat{s}^2$$

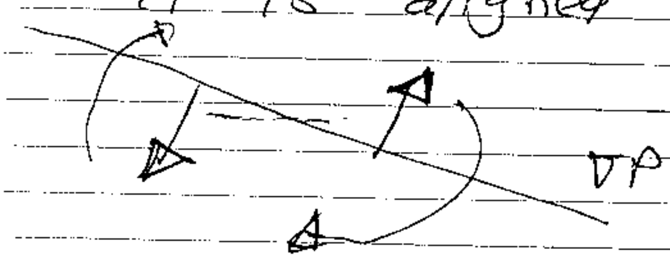
but $1/R_c = \frac{r}{q^2R^2}$

$\Rightarrow$ $\boxed{\frac{r}{L_p}\beta < \hat{s}^2}$ $\rightarrow$ recovers Suydam, up to #.

i.e.

$\Rightarrow$ in periodic system with resonances, shear induces "effective line-tying", of sorts

$\Rightarrow$ physics is penalty in energy to rotate convective cell so that it is aligned with field.



stability is gain of gradient relaxation vs loss due shear-enforced rotation penalty.

$\Rightarrow$ "ideal" MHD consistent with $k_\theta \Delta \sim 1$ choice. Apart from boundary (excluded here by mode localization), ideal MHD is scale free, here, basic smallness parameter is

$$1/S = \eta/a^2 / V_A/a$$

$1/S$ ↳ Lundquist #

η/a^2 ↳ resistive diffusion rate

V_A/a ↳ Alfven rate

For resistive interchange:

$$-\frac{\partial^2}{\partial t^2} \nabla_\perp^2 \vec{\phi} = \frac{B_0 \cdot \nabla}{\rho_0 c} \frac{\partial \hat{J}_z}{\partial t} + \frac{\partial^2 \vec{\phi}}{\partial y^2} \frac{g_{eff}}{4\rho}$$

$$\frac{\partial \hat{J}_z}{\partial t} - \eta \nabla_\perp^2 \hat{J}_z = \frac{c}{4\pi} \underline{B_0} \cdot \underline{\nabla} \left(-\nabla_\perp^2 \vec{\phi}\right)$$

Assume $\eta k_\perp^2 > \gamma$ (verify aposteriori!)

$$\Rightarrow \quad \hat{J}_z = +\frac{c}{4\pi} \frac{B_0 \cdot \nabla}{\eta} \vec{\phi}$$

(not possible in ideal for $k \cdot B \neq 0$)

(electrostatic approximation i.e. $E_\parallel = \cancel{E_\parallel}_{\text{inductive}} + E_{\parallel es}$

$= \eta J_\parallel$)

$\therefore$

$$-\gamma^2 \left(\frac{\partial^2}{\partial x^2} - k_\theta^2\right) \vec{\phi} = \frac{\gamma i B_0 k_\parallel}{\rho_0 c \eta} \frac{c B_0 i k_\parallel \vec{\phi}}{4\pi} - k_\theta^2 \frac{g_{eff}}{4\rho} \vec{\phi}$$

$$\rightarrow \left(\frac{\partial^2}{\partial x^2} - k_\theta^2\right)\hat{\phi} - \frac{k_\parallel^2 V_A^2}{\gamma\eta}\hat{\phi} - \frac{k_\theta^2 g_{eff}}{L_p \gamma^2}\hat{\phi} = 0$$

$k_\parallel = k_\theta x / L_s$

$\Rightarrow$ eigenvalue problem for $\gamma_{m,n}$:

$$\left(\frac{\partial^2}{\partial x^2} - k_\theta^2\right)\hat{\phi}_{m,n} - \frac{k_\theta^2 V_A^2}{L_s^2 \gamma\eta} x^2 \hat{\phi}_{m,n} - \frac{k_\theta^2 g_{eff}}{\gamma^2 L_p}\hat{\phi}_{m,n} = 0$$

Now, $\hat{\phi}_{m,n} = e^{-\alpha_{m,n} x^2/2}$ $\qquad \alpha_{m,n} \sim 1/(\Delta x_{m,n})^2$

$\rightarrow$ "slow" interchange $\quad (k_\theta \Delta x < 1)$

$$\alpha^2 x^2 - \alpha \quad - \frac{k_\theta^2 V_A^2 x^2}{L_s^2 \gamma\eta} - \frac{k_\theta^2 g_{eff}}{\gamma^2 L_p} = 0$$

$$\alpha = \left(\frac{k_\theta^2 V_A^2}{L_s^2 \gamma\eta}\right)^{1/2}$$ $\rightarrow$ defines basic mode scale (η dependent, γ dependent)

$$\alpha = \frac{k_\theta^2 g_{eff}}{\gamma^2 |L_p|}$$ $\rightarrow$ dispersion relation (need $g_{eff}/L_p < 0$

To determine γ, α explicitly :

$$\left(\frac{k_\theta^2 V_A^2}{L_s^2 \gamma \eta}\right)^{1/2} = \frac{k_\theta^2 g_{eff}}{\gamma^2 L_p}$$

$$\Rightarrow \quad \gamma = \left(\frac{L_s^2}{L_p^2}\,(\eta k_\theta^2)\left(\frac{V_A^2}{R_c^2}\right)\beta^2\right)^{1/3}$$

so

$$\begin{cases} \alpha = \left(k_\theta^2 V_A^2 / L_s^2 \left(\frac{L_s^2}{L_p^2}\,\eta k_\theta^2 \,\frac{V_A^2}{R_c^2}\,\beta^2\right)^{1/3} \eta\right)^{1/2} \\ \Delta X = 1/\sqrt{\alpha} \end{cases}$$

For validity:

$$\begin{cases} \frac{\eta}{(\Delta X)^2} = \eta\alpha > \gamma \\ k_\theta^2 (\Delta X)^2 = \frac{k_\theta^2}{\alpha} < 1 \end{cases}$$

$\Rightarrow$ for e.s.:

$$\eta^2 \frac{k_\theta^2 V_A^2}{L_s^2 \gamma \eta} > \gamma^2$$

$$\cancel{\eta}\,\frac{\cancel{k_\theta^2}\cancel{V_A^2}}{L_s^2} > \gamma^3 = \frac{L_s^2}{L_p^2}\,\cancel{\eta}\cancel{k_\theta^2}\,\frac{\cancel{V_A^2}}{R_c^2}\,\beta^2$$

i.e. Need : $\frac{\beta L_s^2}{|L_p| R_c} < 1$ for validity of electrostatic approximation.

Note:

i.) $\left.\begin{matrix} R_c \sim a \\ m/r \sim k_\theta \end{matrix}\right\} \Rightarrow$

$$\gamma \sim \left(\frac{L_s^2 m^2 \beta^2}{L_p^2}\right)^{1/3} \left(\frac{\eta}{a^2} \frac{V_A^2}{a^2}\right)^{1/3}$$

$$\sim \left(1/\tau_R \tau_A^2\right)^{1/3} \Rightarrow \gamma \tau_A \sim S^{-1/3} \beta^{2/3} (L_s/L_p)$$

i.e. growth rate is hybrid of resistive diffusion and Alfven rates

$\Rightarrow$ resistive diffusion allows decoupling of field, fluid, thereby triggering instability.

ii.) for incompressible MHD, have instability for all β (i.e. unlike ideal MHD, no β_{crit} exists!)

iii.) $\Delta x = \left(L_s^2 \gamma \eta / k_\theta^2 V_A^2\right)^{1/4} \ll a$

$$\sim S^{-1/3} \beta^{1/6}$$

i.e. $\Delta x/a \sim S^{-1/3} \leftrightarrow$ narrow layer.

(iv) For fast interchange: $k_{\theta}^2 (\Delta x)^2 > 1$

Thus, as before:

$$\alpha^2 x^2 - \alpha - k_\theta^2 - \frac{k_\theta^2 V_A^2 x^2}{L_s^2 \gamma \eta} - \frac{g_{eff} k_\theta^2}{\gamma^2 L_p} = 0$$

$$\Rightarrow \quad \alpha = \left(k_\theta^2 V_A^2 / L_s^2 \gamma \eta\right)^{1/2}$$

$\frac{k_\theta^2}{\alpha} > 1 \quad \Rightarrow$ now obtain dispersion relation:

$$-k_\theta^2 = -g_{eff} k_\theta^2 / \gamma^2 |L_p|$$

$$\gamma^2 = g_{eff}/|L_p| = c_s^2 / R_c |L_p|$$

$$\Delta x \sim S^{-1/2}$$

Note:

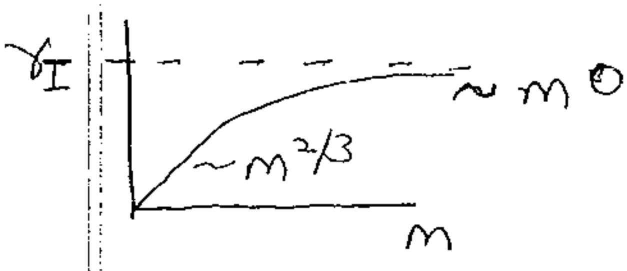
(i) Fast regime entered when:

$$\frac{k_\theta^2}{\alpha} > 1 \quad \Rightarrow \quad \frac{k_\theta^2}{\left(k_\theta^2 V_A^2 / L_s^2 \gamma \eta\right)^{1/2}} > 1$$

$$k_\theta^{\cancel{2}4} > \frac{\cancel{k_\theta^2} V_A^2}{L_s^2 \gamma \eta}$$

$\eta k_\theta^2 > V_A^2/L_s^2 \, \gamma_I$

i.e. fast interchanges dominate at large m



In practice, large η or high $m \Rightarrow$ fast interchange

ii.) Note essence of fast interchange is:

- high ηk_θ^2
- ideal growth rate.

Physical content is that ηk_θ^2 so large that line-bending destroyed and mode reverts to ideal growth

iii.) Note $\Delta x \sim S^{-1/2}$, i.e. mode still localized by η. also $\gamma \Delta^2 \sim \eta$

iv.) In reality, fast interchange eventually cut-off by dissipation (μ, D etc.).

v.) All resistive interchanges localized to $\underline{k} \cdot \underline{B}_0 = 0$ resonant surfaces.